Lemma: Let the graph of the equation $\Phi(\rho,\theta) = 0$ have a branch Γ, a portion of which, Γ_1, extends to infinity. Let Γ_1 be contained in the sector $\theta_1 \leqq \theta \leqq \theta_2$ where $\theta_2 - \theta_1 < 2\pi$. Let $\Phi(\rho,\theta)$ change sign as Γ_1 is crossed, and let there be no other branch in this sector having this property. Then no closed trajectory which encloses the origin can have points in common with Γ_1.

Proof: At points of Γ_1 where $F(\rho,\theta) \neq 0$, the direction field of integral curves is tangent to the radius vector drawn from the origin. Since $\Phi(\rho,\theta)$ changes sign as Γ_1 is crossed, the coordinate θ on the integral curves has an extreme value when the integral curve meets Γ_1. Also, since there is no other branch of $\Phi(\rho,\theta) = 0$ in the sector, on crossing which $\Phi(\rho,\theta)$ changes sign, a trajectory entering this sector will always remain within it. This proves the lemma.

Corollary: If the graph of the equation $\Phi(\rho,\theta) = 0$ has a unique branch in the sector $\theta_1 \leqq \theta \leqq \theta_2$, with one end at the origin and the other extending to infinity, and if $\Phi(\rho,\theta)$ changes sign on crossing this branch, then the origin is acyclic.

Let us now consider the system of differential equations

$$\begin{aligned} dx/dt &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ dy/dt &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{aligned} \tag{3}$$

with constant coefficients. The following theorem holds.

Theorem: In the system (3) a singular point of nodal type cannot be enclosed by a closed trajectory.

Proof: Let us assume that the origin is a singular point of nodal type, and let us introduce a system of polar coordinates. The transformed system has the form

$$\begin{aligned} d\rho/dt = \rho\{&a_{10}\cos^2\theta + (a_{01} + b_{10}) \cos\theta \sin\theta + \\ &b_{01}\sin^2\theta + \rho[a_{20}\cos^3\theta + (a_{11} + b_{20})\cos^2\theta \sin\theta + \\ &(a_{02} + b_{11})\cos\theta \sin^2\theta + b_{02} \sin^3\theta]\} \end{aligned} \tag{4}$$

$$\begin{aligned} d\theta/dt = b_{10} &\cos^2\theta + (b_{01} - a_{10}) \cos\theta \sin\theta - \\ &a_{01} \sin^2\theta + \rho[b_{20}\cos^3\theta + (b_{11} - a_{20}) \times \\ &\cos^2\theta \sin\theta + (b_{02} - a_{11}) \cos\theta \sin^2\theta - \\ &a_{02} \sin^3\theta] \equiv f_2(\theta) + \rho f_3(\theta) \equiv \Phi(\rho,\theta) \end{aligned}$$

Since $x = 0$, $y = 0$ is a singular point of nodal type, $f_2(\theta)$ either vanishes identically or has a real zero, θ_1. The equation $f_3(\theta) = 0$, since it is a homogeneous polynomial of third degree in $\sin\theta$ and $\cos\theta$, always has at least one real root. If $f_2(\theta) \equiv 0$, if $f_3(\theta) \equiv 0$, or if $f_2(\theta_1) = 0$ and $f_3(\theta_1) = 0$, than the conclusion of the theorem follows from the fact that the system (3) has at least one integral curve which is a straight line through the origin. It thus remains to consider the case when the equations $f_2(\theta) = 0$ and $f_3(\theta) = 0$ have no common root. Clearly we can always find a sector (θ_1,θ_2) such that $f_2(\theta_1) = 0$ and $f_3(\theta_2) = 0$, and such that neither $f_2(\theta)$ nor $f_3(\theta)$ vanishes for any other value of θ in this sector. Consider the curve $\rho = -f_2(\theta)/f_3(\theta)$ in this sector. Without loss of generality we may assume that the function $-f_2(\theta)/f_3(\theta)$ is positive within the sector, since if it were negative, we would consider the opposite sector $(\theta_1 \pm \pi, \theta_2 \pm \pi)$, in which it would be positive. Clearly $\rho = -f_2(\theta)/f_3(\theta)$ represents a branch of the graph of $\Phi(\rho,\theta) = 0$ for the system (4), which has one end at the origin and the other extending to infinity, wholly contained in a sector less than 2π. As for the variation of sign of $\Phi(\rho,\theta)$ on crossing this branch, this follows from the linearity of $\Phi(\rho,\theta)$ in ρ. There are no other branches of $\Phi(\rho,\theta) = 0$ in this sector. Thus the conditions of the corollary to the lemma are satisfied. The theorem is proved.

Taking into account the fact that the system (3) can have no more than two singular points of "focus" type,[1] we obtain, as a corollary to the theorem just proved, a known result:[2] If the system (3) has three limit cycles, $L_c^i (i = 1,2,3)$, then it is impossible for $I_c^i \cap I_c^j = 0$ for $i \neq j,i,j = 1,2,3$. (I_c^k denotes the domain enclosed by L_c^k.)

We mention in conclusion that this problem was proposed in the seminar of N. P. Erugin and Yu. S. Bogdenov. The forementioned example was presented there.

—*Submitted March 1, 1960*

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Convergence of a Generalized Interpolation Polynomial

V. N. Rusak

In his paper[1] S. N. Bernshtein proved that the generalized polynomial which deviates least from zero on the interval $[-1,1]$ among all generalized polynomials of the form

$$\frac{x^n + b_1x^{n-1} + \ldots + b_n}{\prod_{i=1}^{n}\left(1 - \frac{x}{a_i}\right)} = A_0 + \sum_{k=1}^{n} \frac{A_k}{a_k - x}$$

has the representation

$$r_n^*(x) = C \cos(n\theta + \psi) \tag{1}$$

where C is a definite constant

$$\cos\theta = x \qquad \psi = 2 \sum_{k=1}^{n} \alpha_k$$

$$\cos\alpha_k = \frac{\rho_k - x}{\sqrt{2\rho_k(a_k - x)}}$$

$$\sin\alpha_k = \frac{\sqrt{1 - x^2}}{\sqrt{2\rho_k(a_k - x)}}$$

$$a_k = \frac{1}{2}\left(\rho_k + \frac{1}{\rho_k}\right)$$

$$|\rho_k| > 1 \qquad k = 1, 2, \ldots, n$$

We consider generalized interpolation polynomials on the interval $[-1,1]$ with respect to the system of Chebyshev functions

$$\frac{1}{a_1 - x}, \quad \frac{1}{a_2 - x}, \ldots, \quad \frac{1}{a_n - x} \tag{2}$$

$$|a_i| > 1 \qquad i = 1, 2, \ldots, n$$

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that is, polynomials of the form

$$R_n(x,f) = \sum_{k=1}^{n} \frac{A_k}{a_k - x} = \frac{\sum_{k=1}^{n-1} b_k x^k}{\prod_{i=1}^{n} \left(1 - \frac{x}{a_i}\right)} \tag{3}$$

which satisfy conditions $R_n(x_\nu,f) = f(x_\nu)$, where $\{x_\nu = \cos\theta_\nu\}_1^n$ are the zeros of the polynomial (1)* ($\nu = 1, 2, \ldots, n$).

We obtain the following result.

Theorem: Let α_0 be a fixed number, and let

$$\frac{|\rho_k| - 1}{|\rho_k| + 1} \geqq \alpha_0 \qquad 0 < \alpha_0 < 1 \qquad k = 1, 2, \ldots, n \tag{4}$$

Then for continuous functions on $[-1,1]$ *and any* n *we have*

$$|f(x) - R_n(x,f)| < E_n(f)\left(\frac{2\pi^2}{\alpha_0^4} + \frac{4}{\alpha_0^2}\ln\alpha_0 n + 1\right)$$

where $E_n(f)$ is the best approximation of function f by polynomials of system (2) *on the interval* $[-1,1]$.

Corollary: Under condition (4)† the generalized interpolation polynomials converge uniformly to functions continuous on $[-1,1]$ if $E_n(f)\ln n \to 0$ as $n \to \infty$.

Proof of the Theorem: Taking into account the representation (1) the Bernshtein interpolation polynomial (3) will have the form

$$R_n(x,f) = \sum_{k=1}^{n} f(x_k)\frac{\cos(n\theta + \psi)}{(x - x_k)r_n'(x_k)}$$

where $r_n(x) = \cos(n\theta + \psi)$.

Our theorem will be a direct consequence of Lebesgue's theorem[4] if we show that

$$\|R_n(x,f)\| = \max_{-1\leqq x\leqq 1} \sum_{k=1}^{n} \left|\frac{\cos(n\theta + \psi)}{(x - x_k)r_n'(x_k)}\right| \leqq \frac{2\pi^2}{\alpha_0^4} + \frac{4}{\alpha_0^2}\ln\alpha_0 n \tag{5}$$

We first obtain

$$r_n'(x) = \frac{\sin(n\theta + \psi)}{\sqrt{1 - x^2}}\left(n + \sum_{k=1}^{n} \frac{\rho_k x - 1}{\rho_k(a_k - x)}\right) = \frac{\sin(n\theta + \psi)}{\sin\theta}\sum_{k=1}^{n}\frac{\rho_k a_k - 1}{\rho_k(a_k - x)} = \sin(n\theta + \psi)\frac{\varphi(x)}{\sin\theta}$$

$$|r_n'(x_k)| = \frac{\varphi(x_k)}{\sin\theta_k} \tag{6}$$

We now estimate function $\varphi(x)$ on interval $[-1,1]$:

$$|\varphi(x)| = \varphi(x) = \sum_{k=1}^{n} \frac{\rho_k a_k - 1}{\rho_k(a_k - x)} \leqq \sum_{k=1}^{n} \frac{\rho_k a_k - 1}{|\rho_k|\{|a_k| - 1\}} = \sum_{k=1}^{n} \frac{|\rho_k| + 1}{|\rho_k| - 1}$$

$$\varphi(x) \geqq \sum_{k=1}^{n} \frac{\rho_k a_k - 1}{|\rho_k|\{|a_k| + 1\}} = \sum_{k=1}^{n} \frac{|\rho_k| - 1}{|\rho_k| + 1}$$

that is, by condition (4)

$$\alpha_0 n \leqq \varphi(x) \leqq n/\alpha_0 \tag{7}$$

By virtue of (6) and (7) we obtain the following bound for the norm:

$$\|R_n(x,f)\| = \max_{-1\leqq x\leqq 1} |\cos(n\theta + \psi)| \sum_{k=1}^{n} \frac{\sin\theta_k}{|x - x_k|\varphi(x_k)} \leqq \max_{0\leqq\theta\leqq\pi} \frac{|\cos(n\theta + \psi)|}{n\alpha_0} \sum_{k=1}^{n} \frac{\sin\theta_k}{|\cos\theta - \cos\theta_k|}$$

From the obvious relation

$$|n\theta_{k+1} + \psi(\theta_{k+1}) - n\theta_k - \psi(\theta_k)| = \pi$$

we obtain, taking (7) into account and the fact that

$$\frac{d}{d\theta}(n\theta + \psi) = \varphi(x)$$

the bounds

$$\frac{\alpha_0\pi}{n} \leqq |\theta_{k+1} - \theta_k| \leqq \frac{\pi}{\alpha_0 n} \qquad k = 1, 2, \ldots, n$$

The further estimate (5) is obtained in the same manner as in the case of algebraic interpolation with respect to Chebyshev nodes (cf. Ref. 3).

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* Polynomial (1) has n distinct zeros on the interval $[-1,1]$ by virtue of Chebyshev's theorem on polynomials with least deviation.

† Condition (4) is more restrictive than the criterion of closure of system (2) in $C\,[-1,1]$ (cf. Ref. 2).